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A Hopf-Like Equation and Perturbation Theory for Differential Delay Equations

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We extend techniques developed for the study of turbulent fluid flows to the statistical study of the dynamics of differential delay equations. Because the phase spaces of differential delay equations are infinite dimensional, phase-space densities for these systems are functionals. We derive a Hopf-like functional differential equation governing the evolution of these densities. The functional differential equation is reduced to an infinite chain of linear partial differential equations using perturbation theory. A necessary condition for a measure to be invariant under the action of a nonlinear differential delay equation is given. Finally, we show that the evolution equation for the density functional is the Fourier transform of the infinite-dimensional version of the Kramers–Moyal expansion.

KEY WORDS: Hopf-like equation; perturbation theory; differential delay equations.

1. INTRODUCTION

We derive a functional differential equation for the characteristic functional $\mathscr{Z}_t$ of the measure defined on the phase space of a nonlinear differential delay equation (DDE). This functional equation describes the evolution of a density of initial functions under the action of a DDE. We show that the evolution equation for $\mathscr{Z}_t$ (a Hopf-like equation) is the Fourier transform of the infinite-dimensional extension of the Kramers–Moyal (KM) expansion. This approach to the study of delayed dynamics was inspired by the work of Capiński,[1] which extended functional techniques introduced by Hopf.[2]

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The formalism used throughout this paper is that of probability theory in function spaces. Consequently, there is a strong analogy between our presentation, field theories, and the functional description of fluid mechanics. In particular, perturbation theory and the expansion of characteristic functionals in terms of probability moments are applicable to the study of differential delay equations.

The outline of the paper is as follows. The characteristic functional $\mathscr{Z}_t$ is introduced in Section 2. The evolution equation for $\mathscr{Z}_t$ is then derived for DDEs with a smooth feedback nonlinearity.

A power series expansion of $\mathscr{Z}_t$ is presented in Section 3 and used to reduce the Hopf-like equation of Section 2.1 to an infinite number of coupled linear hyperbolic partial differential equations (PDEs). We also give necessary conditions to be met by the invariant measure(s) for a nonlinear DDE with a quadratic nonlinearity.

Finally, in Section 4, the connection between the Hopf equation and the functional extension of the Kramers–Moyal expansion is presented.

2. CHARACTERISTIC FUNCTIONALS FOR DELAY EQUATIONS

We consider DDEs of the form

$$\frac{du}{ds} = -\alpha u(s) + F(u(s-1)) \qquad \text{for} \quad 1 < s \tag{1a}$$

in which the delay τ is taken to be 1 without loss of generality, with the initial function

$$u(s) = v(s) \qquad \text{if} \quad s \in [0, 1] \tag{1b}$$

From now on we write (1a), (1b) as the combined system

$$\begin{aligned} u(s) &= v(s) && \text{for} \quad s \in [0, 1] \\ \frac{du(s)}{ds} &= -\alpha u(s) + F(v(s-1)) && \text{for} \quad 1 < s < 2 \end{aligned} \tag{2}$$

and denote by $\mathscr{S}_t$ the corresponding semidynamical system $\mathscr{S}_t$: $\mathscr{C}([0, 1]) \mapsto \mathscr{C}([0, 1])$ given by

$$\mathscr{S}_t v(x) = u_v(x + t) \tag{3}$$

where u_v denotes the solution of (2) corresponding to the initial function v. Equation (3) defines a *semidynamical system* because a DDE is *noninvertible*, i.e., it cannot be run unambiguously forward and backward in time.

From (3), the system (2) is equivalent to considering

$$\frac{\partial}{\partial t}\mathscr{S}_t v(x) = \begin{cases} (\partial/\partial t) v(x+t) & \text{for} \quad x \in [0, 1-t] \\ -\alpha u(x+t) + F(v(x+t-1)) & \text{for} \quad x \in (1-t, 1] \end{cases} \tag{4}$$

Thus, we consider a segment of a solution of (2) defined on an interval $I_t = [t, t+1]$, as t increases (continuously) [i.e., the DDE(2) operates on a *buffer* of length 1, "sliding" it along the time axis]. Equation (4) states that the content of this buffer is the initial condition v when the argument $(x+t)$ is less than 1, and the solution u of the equation otherwise.

We next introduce the characteristic functional $\mathscr{Z}_t$ of a family of probability measures evolving from an initial measure. We define the characteristic functional $\mathscr{Z}_t$ for (4) by

$$\mathscr{Z}_t[J_1, J_2] = \int_{\mathscr{C}} \exp\left[i \int_0^1 J_1(x)\, u_v(x+t)\, dx + i \int_0^1 J_2(x)\, v(x)\, dx \right] \times d\mu_0(\mathscr{T}_t^{-1}(v, u_v)) \tag{5a}$$

The source functions J_1 and J_2 are elements of $\mathscr{C}([0, 1])$ and the measure of integration is the initial measure μ_0 (describing the initial distribution of functions) composed with $\mathscr{T}_t^{-1}(v, u_v)$, where $\mathscr{T}_t(v)$: $\mathscr{C}([0, 1]) \mapsto \mathscr{C} \times \mathscr{C}$ is defined by

$$\mathscr{T}_t(v) = (v, u_v) \tag{5b}$$

For simplicity, we will use the notation $\mu_0(\mathscr{T}_t^{-1}(v, u_v)) \equiv \mathscr{W}[v, \mathscr{S}_t(v)]$, so (5a) becomes

$$\mathscr{Z}_t[J_1, J_2] = \int_{\mathscr{C}} \exp\left[i \int_0^1 J_1(x)\, u_v(x+t)\, dx + i \int_0^1 J_2(x)\, v(x)\, dx \right] \times d\mathscr{W}[v, \mathscr{S}_t(v)] \tag{5c}$$

When no confusion is possible, we write $\mathscr{W}_t$ for $\mathscr{W}[v, \mathscr{S}_t(v)]$.

If f and g are two functions defined on an interval I, we denote their scalar product by

$$\{f, g\} \equiv \int_I f(x)\, g(x)\, dx$$

To simplify the notation, we also write

$$\Upsilon[J_1, J_2; v] = \exp[i\{J_1(x), u_v(x+t)\} + i\{J_2(x), v(x)\}] \tag{6}$$

Υ is used from now on to denote the function of J_1, J_2, and v defined in (6). We begin by noting the following relations:

$$\frac{\delta^n \mathscr{Z}_t}{\delta J_1^n(\xi)} = i^n \langle \Upsilon u_v^n(\xi + t) \rangle \tag{7}$$

$$\frac{\delta^n \mathscr{Z}_t}{\delta J_2^n(\xi)} = i^n \langle \Upsilon v^n(\xi) \rangle \tag{8}$$

where it is understood that

$$\langle (\because) \rangle = \int_{\mathscr{C}} (\because)\, d\mathscr{W}[v, \mathscr{S}_t(v)]$$

Note that if μ_0 is the probability measure on the space of all initial functions v, and A is any subset of $\mathscr{C}([0, 1])$, then

$$\mu_t(A) \equiv \mu_0(\mathscr{S}_t^{-1}(A)) \tag{9}$$

In other words, the probability that a randomly chosen function belongs to A at time t equals the probability that the counterimage of the function (under the action of $\mathscr{S}_t$) belonged to the counterimage of the set A. This defines the family of measures characterized by the solutions $\mathscr{Z}_t$ of a functional differential equation which is the Fourier transform of the infinite-dimensional version of the KM equation. The derivation of such an equation for a DDE was first considered by Capiński.[1] If the semiflow $\mathscr{S}_t$ is measure-preserving with respect to μ_0, then $\mu_0(A) \equiv \mu_0(\mathscr{S}_t^{-1}(A))$. In this case, we alternately say that the measure μ_0 is invariant with respect to $\mathscr{S}_t$.

We are now in a position to derive an evolution equation for the characteristic functional.

2.1. A Functional Differential Equation for $\mathscr{Z}_t$

Time differentiation of the characteristic functional $\mathscr{Z}_t$ defined in (5c) yields, in conjunction with (4),

$$\begin{aligned}
\frac{\partial \mathscr{Z}_t}{\partial t} &= i \left\langle \Upsilon \int_0^1 J_1(x) \frac{\partial u_v(x+t)}{\partial t}\, dx \right\rangle \\
&= i \left\langle \Upsilon \int_0^1 J_1(x) \frac{\partial u_v(x+t)}{\partial x}\, dx \right\rangle \\
&= i \left\langle \Upsilon \int_0^{1-t} J_1(x) \frac{\partial v(x+t)}{\partial x}\, dx - \alpha \Upsilon \int_{1-t}^1 J_1(x)\, u(x+t)\, dx \right\rangle \\
&\quad + i \left\langle \Upsilon \int_{1-t}^1 J_1(x)\, F(v(x+t-1))\, dx \right\rangle
\end{aligned} \tag{10}$$

Therefore, from Eq. (7) and the definition (4), we obtain

$$\frac{\partial \mathscr{Z}_t}{\partial t} = \int_0^{1-t} J_1(x) \frac{\partial}{\partial x} \left(\frac{\delta \mathscr{Z}_t}{\delta J_1(x)} \right) dx - \alpha \int_{1-t}^{1} J_1(x) \frac{\delta \mathscr{Z}_t}{\delta J_1(x)} dx$$
$$+ i \left\langle \Upsilon \int_{1-t}^{1} J_1(x) F(v(x+t-1)) \, dx \right\rangle \tag{11}$$

Equation (11) is related to the Hopf functional differential equation for the evolution of the characteristic functional $\mathscr{Z}_t$, and contains all the statistical information describing the evolution of a density of initial functions under the action of the differential delay system (1a), (1b). An equation similar to (11) was first obtained by Capiński for a differential delay equation with a quadratic nonlinearity (see Example 1 below).

In order to derive the Hopf equation per se, we restrict our attention to situations where the feedback function F in the DDE (1a) is a polynomial

$$F(v) = \sum_{k=1}^{n} a_k v^k \tag{12}$$

With nonlinearity (12), Eq. (11) becomes, with identity (8),

$$\frac{\partial \mathscr{Z}_t}{\partial t} = \int_0^{1-t} J_1(x) \frac{\partial}{\partial x} \left(\frac{\delta \mathscr{Z}_t}{\delta J_1(x)} \right) dx - \alpha \int_{1-t}^{1} J_1(x) \frac{\delta \mathscr{Z}_t}{\delta J_1(x)} dx$$
$$+ \sum_{k=1}^{n} i^{(1-k)} a_k \int_{1-t}^{1} J_1(x) \frac{\delta^k \mathscr{Z}_t}{\delta J_2^k(x+t-1)} dx \tag{13}$$

Analytically solving the Hopf equation (13) is not possible at present, though a correct method of solution should make use of integration with respect to measures defined on the space $\mathscr{C}$. Presently, the theory of such integrals only allows their consistent utilization in solving functional differential equations when the measure of integration is the *Wiener measure.*[13]

The lack of a formalism within which to evaluate functional integrals with respect to general measures also poses a problem for the development of field theories in physics where the characteristic and generating functionals (CF, GF) both play fundamental roles.[3]

In statistical physics, the GF is interpreted as the partition function for systems with an infinite number of degrees of freedom, while in quantum

[3] The *characteristic functional*, presented here, is the *Fourier transform* of a probability distribution (i.e., by Bochner's theorem it is the Fourier transform of a measure[10]). The *generating functional*, however, is the Laplace transform of a probability distribution.

field theory the CF is used to obtain the Green's functions from which the scattering amplitudes for various processes are calculated. In quantum field theory, the measure of integration is the Wiener measure for the free-particle problem, for which the field equations are Wiener-measure-preserving; this is not the case when particles interact, and the systems under consideration are no longer Wiener-measure-preserving. In that case, investigators reduce the functional integral to a countably infinite product of finite-dimensional integrals by coarse-graining the phase space (or, in the language of quantum field theory, replacing the continuum by a lattice.[12]

Before proceeding to treat the Hopf equation in a perturbative manner, we illustrate its specific form for a simple nonlinear delay equation.

Example 1. The differential delay equation

$$\frac{du}{ds} = -\alpha u(s) + ru(s-1)[1-u(s-1)] \tag{14}$$

can be considered as a continuous analogue of the discrete-time quadratic map

$$u_{n+1} = ru_n(1-u_n) \tag{15}$$

because Eq. (14) is the *singular perturbation* of the quadratic map (15) as defined in ref. 3. The characteristic functional is defined by (5c), and the functional differential equation corresponding to (13) was shown by Capiński[1] to be

$$\begin{aligned} \frac{\partial \mathscr{Z}_t}{\partial t} &= \int_0^{1-t} J_1(x) \frac{\partial}{\partial x}\left(\frac{\delta \mathscr{Z}_t}{\delta J_1(x)}\right) dx - \alpha \int_{1-t}^{1} J_1(x) \frac{\delta \mathscr{Z}_t}{\delta J_1(x)}\, dx \\ &\quad + r \int_{1-t}^{1} J_1(x) \frac{\delta \mathscr{Z}_t}{\delta J_2(x+t-1)}\, dx - ri^{-1} \int_{1-t}^{1} J_1(x) \frac{\delta^2 \mathscr{Z}_t}{\delta J_2^2(x+t-1)}\, dx \end{aligned} \tag{16}$$

In spite of the fact that we cannot solve the Hopf equation analytically, relatively mild assumptions allow us to gain significant insight into the dynamics of $\mathscr{Z}_t$. More precisely, if $\mathscr{Z}_t$ is analytic, we can expand it in a power series and treat the Hopf equation in a perturbative manner. We follow this approach in the next section.

3. THE MOMENTS OF THE MEASURE $\mathscr{W}_t$

The statistical properties of the random field of functions v and u are described by an infinite hierarchy of moments of the measure $\mathscr{W}_t$. For fixed

t, the average value of the contents of the buffer defined on $I_t = [t, t+1]$ (i.e., v on $[t, 1]$ and u_v on $(1, 1+t]$), which is just the first-order moment of the measure $\mathscr{W}_t$, is

$$M_v^1(t, x) \equiv \int_{\mathscr{C}} v(x+t)\, d\mu_0(v) \qquad \text{for} \quad x \in [0, 1-t] \tag{17}$$

$$M_u^1(t, x) \equiv \int_{\mathscr{C}} u_v(x+t)\, d\mu_0(u) \qquad \text{for} \quad x \in (1-t, 1] \tag{18}$$

These two equations can be written as one relation:

$$M^1(t, x) \equiv \int_{\mathscr{C}} u_v(x+t)\, d\mathscr{W}_t \qquad \text{for} \quad x \in [0, 1] \tag{19}$$

The definition of the second-order moment (or covariance function) $M^2(t, x, y)$ is, with the same notation,

$$\begin{aligned}
M^2(t, x, y) &= \int_{\mathscr{C}} v(x+t)\, v(y+t)\, d\mathscr{W}_t \\
&\equiv M_{vv}^2(t, x, y) \qquad \text{for} \quad x, y \in [0, 1-t] \times [0, 1-t] \\
M^2(t, x, y) &= \int_{\mathscr{C}} u_v(x+t)\, v(y+t)\, d\mathscr{W}_t \\
&\equiv M_{uv}^2(t, x, y) \qquad \text{for} \quad x, y \in (1-t, 1] \times [0, 1-t] \\
M^2(t, x, y) &= \int_{\mathscr{C}} v(x+t)\, u_v(y+t)\, d\mathscr{W}_t \\
&\equiv M_{vu}^2(t, x, y) \qquad \text{for} \quad x, y \in [0, 1-t] \times (1-t, 1] \\
M^2(t, x, y) &= \int_{\mathscr{C}} u_v(x+t)\, u_v(y+t)\, d\mathscr{W}_t \\
&\equiv M_{uu}^2(t, x, y) \qquad \text{for} \quad x, y \in (1-t, 1] \times (1-t, 1]
\end{aligned}$$

The subscripts of the various components of M^2 refer to the segments of the solution whose correlation is given by the particular component. For example, M_{uv}^2 describes the correlation between u and v segments of the solution as illustrated in Fig. 1. Remember that the initial function is defined on an interval [0, 1], so to complete the description of the statistical dependence of the solution u_v on the initial function it is necessary to introduce the functions M_{ou}^2. M_o^1 is the first-order moment of measure μ_0, M_{oo}^2 is the second order moment of μ_0, etc.

The moments of the measure $\mathscr{W}_t$ are also given by the power series expansion of the characteristic functional $\mathscr{Z}_t$, as we next discuss.

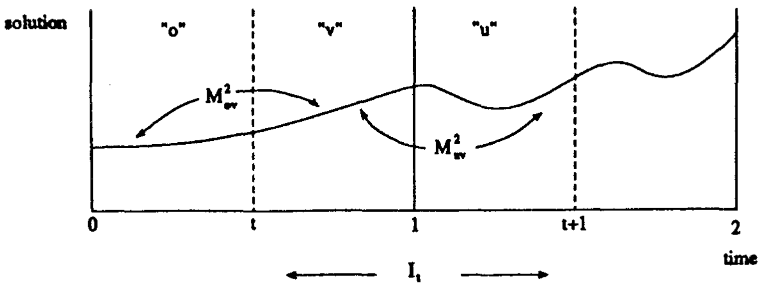

Fig. 1. A DDE transforms a function defined on $[0, 1]$ into a function defined on I_t. Illustration of the o, v, and u segments of the solution.

3.1. Taylor Series Expansion of the Functional $\mathscr{L}_t$

The expression for the series expansion of a functional can be understood with the following argument. Let

$$F(y_1,..., y_k) = F(\mathbf{y})$$

be a function of k variables. The power series expansion of F is

$$F(\mathbf{y}) = \sum_{n=0}^{\infty} \sum_{i_1=0}^{k} \cdots \sum_{i_n=0}^{k} \mathscr{E}_n(i_1,..., i_n)(y_1,..., y_n) \tag{20}$$

where

$$\mathscr{E}_n(i_1,..., i_n) = \left.\frac{\partial^n F(\mathbf{y})}{\partial y_1 \cdots \partial y_n}\right|_{\mathbf{y}=0}$$

Passing to a continuum in the sense

$$\begin{gathered} i \to x_i \\ y_i(i = 1,..., k) \to y(x) \\ -\infty < x < \infty \\ \sum_i \to \int_{\mathbf{R}} dx \end{gathered} \tag{21}$$

we obtain the corresponding series expansion of a *functional* $\mathscr{F}$:

$$\mathscr{F}[y] = \sum_{n=0}^{\infty} \int_{\mathbf{R}^n} dx_1 \cdots dx_n\, \mathscr{E}_n(x_1,..., x_n)\, y(x_1) \cdots y(x_n) \tag{22}$$

where

$$\mathscr{E}_n(x_1,\ldots,x_n) = \frac{1}{n!} \frac{\delta^n \mathscr{F}[y]}{\delta y(x_1)\cdots\delta y(x_n)}\bigg|_{y=0} \tag{23}$$

$\mathscr{F}[y]$ is called the *characteristic functional* of the functions $\mathscr{E}_n$.

With these conventions, the expansion of the characteristic functional (5c) is

$$\begin{aligned}\mathscr{Z}_t[J_1, J_2] = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \int_0^1 \cdots \int_0^1 \mathscr{E}_{pq}(t, x_1,\ldots, x_p)\\ \times\left(\prod_{j=1}^{q} J_1(x_j)\,dx_j\right)\left(\prod_{j=q+1}^{p} J_2(x_j)\,dx_j\right)\end{aligned} \tag{24}$$

The kernels $\mathscr{E}_{pq}$ in the expansion are proportional to the moment functions of the measure $\mathscr{W}[v, \mathscr{S}_t v]$. From Eqs. (7) and (8) they are given by

$$\begin{aligned}\mathscr{E}_{pq}(t, x_1,\ldots, x_p) &= \frac{1}{p!} \frac{\delta^p \mathscr{Z}_t}{\delta J_1^q\, \delta J_2^{p-q}}\\ &= \frac{i^p}{p!} \langle u_v(x_1)\cdots u_v(x_q)\, v(x_{q+1})\cdots v(x_p)\rangle \qquad (25)\\ &= \frac{i^p}{p!} M^p_{u^q v^{(p-q)}}(t, x_1,\ldots, x_p) \qquad (26)\end{aligned}$$

where from now on we use the notation

$$M^p_{u^q v^{(p-q)}}(t, x_1,\ldots, x_p) = M^p_{u^q v^{(p-q)}}(t, \mathbf{x})$$

Equation (24) is the infinite-dimensional generalization of the well-known expansion of a characteristic function in terms of the corresponding probability moments (or their Legendre transforms, the cumulants).

3.2. PDEs for the Moments

The evolution equation of the kth moment is given by substituting the moment in question into (13) and then using formula (24) to the appropriate order.

Consider the first-order moments of the measure $\mathscr{W}_t$. If we substitute the definitions (25)–(26) and the expansion (24) into Eq. (13), we obtain a PDE for the moment $M^1(t, x)$:

$$\frac{\partial}{\partial t} M_v^1(t, x) = \frac{\partial}{\partial x} M_v^1(t, x) \qquad \text{for} \quad x \in [0, 1-t]$$

$$\frac{\partial}{\partial t} M_u^1(t, x) = -\alpha M_u^1(t, x) + \sum_{k=1}^{n} a_k M_{o^k}^k(x+t-1, \overset{k}{\ldots}, x+t-1) \qquad (27)$$

$$\text{for} \quad x \in (1-t, 1]$$

Equation (27) is simply the Hopf equation (13) for the first-order moments. In (27) the k arguments of $M_{o^k}^k$ indicate that it is the k-point autocorrelation function of the initial function distribution described by μ_0. Moments whose labels do not contain u are moments of the initial measure.

The second-order moment functions $M^2(t, x)$ are given by the solutions of the four equations

$$\frac{\partial}{\partial t} M_{vv}^2(t, x, y) = \frac{\partial}{\partial x} M_{vv}^2(t, x, y) + \frac{\partial}{\partial y} M_{vv}^2(t, x, y)$$
$$\text{for} \quad (x, y) \in [0, 1-t) \times [0, 1-t) \qquad (28)$$

$$\frac{\partial}{\partial t} M_{uv}^2(t, x, y) = \frac{\partial}{\partial y} M_{uv}^2(t, x, y) - \alpha M_{uv}^2(t, x, y)$$
$$+ \sum_{k=2}^{n} a_k M_{o^{(k-1)}v}^k(t, x+t-1, \overset{(k-1)}{\ldots}, x+t-1, y)$$
$$\text{for} \quad (x, y) \in (1-t] \times [0, 1-t] \qquad (29)$$

$$\frac{\partial}{\partial t} M_{vu}^2(t, x, y) = \frac{\partial}{\partial x} M_{vu}^2(t, x, y) - \alpha M_{vu}^2(t, x, y)$$
$$+ \sum_{k=2}^{n} a_k M_{vo^{(k-1)}}^k(t, x, y+t-1, \overset{(k-1)}{\ldots}, y+t-1)$$
$$\text{for} \quad (x, y) \in [0, 1-t] \times (1-t, 1] \qquad (30)$$

$$\frac{\partial}{\partial t} M_{uu}^2(t, x, y) = -2\alpha M_{uu}^2(t, x, y)$$
$$+ \sum_{k=1}^{n} a_k \{ M_{o^{(k-1)}u}^k(t, x+t-1, \overset{(k-1)}{\ldots}, x+t-1, y)$$
$$+ M_{uo^{(k-1)}}^k(t, x, y+t-1, \overset{(k-1)}{\ldots}, y+t-1) \}$$
$$\text{for} \quad (x, y) \in (1-t, 1] \times (1-t, 1] \qquad (31)$$

The functions M^2_{ou} and M^2_{oou} are given by

$$\frac{\partial}{\partial t} M^2_{ou}(t, x, y) = -\alpha M^2_{ou} + \sum_{k=2}^{n} a_k M^k_{o^k}(x, y+t-1, \overset{k}{\ldots}, y+t-1) \tag{32}$$

$$\frac{\partial}{\partial t} M^3_{oou}(t, x, y, z) = -\alpha M^3_{oou}(t, x, y, z) + \sum_{k=3}^{n} a_k M^k_{o^k}(x, y, z+t-1, \overset{k}{\ldots}, z+t-1) \tag{33}$$

and similar equations give the moments $M^k_{o^{(k-1)}u}$.

A pattern clearly emerges from the preceding analysis: The moment $M^p(t, \mathbf{x})$ is given by 2^p partial differential equations of the same form as (28)-(31) since $M^p(t, \mathbf{x})$ is a function of p variables, each of which can belong to one of two possible intervals ($[0, 1-t]$ or $(1-t, 1]$). The first of these equations (when all the x_k belong to $[0, 1-t]$) is

$$\frac{\partial}{\partial t} M^p_{v^p}(t, \mathbf{x}) = \sum_{j=1}^{p} \frac{\partial}{\partial x_j} M^p_{v^p}(t, \mathbf{x}) \tag{34}$$

We call the equations which give the moments of the form $M^p_{v^l u^{(p-l)}}$ *mixed* equations because they yield functions which correlate u and v segments of the solution. For the moment of order p, there are (2^p-2) mixed equations and 2 *pure* equations. The pure equations give $M^p_{v^p}$ and $M^p_{u^p}$, the p-point autocorrelation functions of the v and u segments of the solution.

If $x_j \in [0, 1-t]$ for $j = 1, \ldots, l$ and $x_j \in (1-t, 1]$ for $j = l+1, \ldots, p$, then when the forcing term F of Eq. (1a) is the polynomial (12), the generic form of the mixed equation for $M_{v^l u^{(p-l)}}$ is

$$\frac{\partial}{\partial t} M^p_{v^l u^{(p-l)}}(t, \mathbf{x}) = \sum_{i=1}^{l} \frac{\partial}{\partial x_i} M^p_{v^l u^{(p-l)}}(t, \mathbf{x}) - \alpha(p-l)\, M^p_{v^l u^{(p-l)}}(t, \mathbf{x}) + \sum_{j=0}^{n-1} a_j \left\{ M^{(p+j)}_{v^l o^j u^{(p-l)}}(t, \mathbf{x}) + M^{(p+j)}_{v^l u^{(p-l)} o^j}(t, \mathbf{x}) \right\} \tag{35}$$

Once again, this equation is one representative of the (2^p-2) mixed equations to be solved to obtain the moment of order p. Deriving these equations is tedious, but the task is greatly simplified by the similarity existing between the systems of equations for moments of different orders.

Equation (24) presented above is reminiscent of functional expansions in quantum field theory and statistical mechanics, which are usually dealt with using Feynman diagrams.

In quantum field theory, Feynman diagrams are used to represent the terms in the expansion of a characteristic functional which describes the distribution of fields (in physics, fields are elements of a function space: they are functions, or *paths* in the phase space). The evolution equations for these fields are obtained by replacing various Lagrangians in the Euler–Lagrange equations which result from applying the principle of least action.[12] Feynman diagrams are used to represent the moments (or n-point correlation functions) of the distribution of fields. The nth moment of the distribution (or n-point correlation function) is represented by n diagrams.[9] In our treatment of delayed dynamics, the nth moment can also be represented by graphs. Preliminary studies indicate that a graphical treatment of (24) is possible, but we have been unable so far to make significant progress. We leave as an open problem the efficient use of Feynman diagrams for the probabilistic description of delayed dynamics.

Before proceeding, we illustrate the ideas presented above and derive the partial differential equations analogous to (27) and (28)–(31) for the nonlinear DDE (14) considered in Example 1.

Example 2. When the DDE is

$$\frac{du}{ds} = -\alpha u(s) + ru(s-1) - ru^2(s-1) \tag{36}$$

the first-order moment equations are given by

$$\frac{\partial M_v^1(t, x)}{\partial t} = \frac{\partial M_v^1(t, x)}{\partial x} \tag{37}$$

$$\frac{\partial M_u^1(t, x)}{\partial t} = -\alpha M_u^1(t, x) + rM_o^1(x+t-1) - rM_{oo}^1(x-t-1, x+t-1) \tag{38}$$

The four evolution equations for the second-order moments are

$$\frac{\partial M_{vv}^2(t, x, y)}{\partial t} = \frac{\partial M_{vv}^2(t, x, y)}{\partial x} + \frac{\partial M_{vv}^2(t, x, y)}{\partial y} \qquad \text{for} \quad x, y \in [0, 1-t] \tag{39}$$

$$\begin{aligned}\frac{\partial M_{vu}^2(t, x, y)}{\partial t} = {} & \frac{\partial M_{vu}^2(t, x, y)}{\partial x} - \alpha M_{vu}^2(t, x, y) + rM_{vo}^2(t, x, y+t-1) \\ & - rM_{voo}^3(t, x, y+t-1, y+t-1) \\ & \text{for} \quad x \in [0, 1-t], \quad y \in (1-t, 1]\end{aligned} \tag{40}$$

$$\frac{\partial M^2_{uv}(t,x,y)}{\partial t}=\frac{\partial M^2_{uv}(t,x,y)}{\partial y}-\alpha M^2_{uv}(t,x,y)+rM^2_{ov}(t,x+t-1,y)$$
$$-rM^3_{oov}(t,x+t-1,x+t-1,y)$$
$$\text{for}\quad x\in(1-t,1],\quad y\in[0,1-t] \tag{41}$$

$$\frac{\partial M^2_{uu}(t,x,y)}{\partial t}=-2\alpha M^2_{uu}(t,x,y)$$
$$+r[M^2_{ou}(t,x+t-1,y)+M^2_{uo}(t,x,y+t-1)]$$
$$-r[M^3_{oou}(t,x+t-1,x+t-1,y)$$
$$+M^3_{uoo}(t,x,y+t-1,y+t-1)]$$
$$\text{for}\quad x,y\in(1-t,1] \tag{42}$$

To solve these equations, one needs to solve first for the moments M^2_{ou}, M^2_{uo}, and M^3_{oou}, which satisfy equations of the form

$$\frac{\partial M^2_{ou}(t,x,y)}{\partial t}=-\alpha M^2_{ou}(t,x,y)+rM^2_{oo}(t,x,y)-rM^3_{ooo}(t,x,y,z) \tag{43}$$

$$\frac{\partial M^3_{oou}(t,x,y,z)}{\partial t}=-\alpha M^3_{oou}(t,x,y,z)+rM^3_{ooo}(x,y,z+t-1)$$
$$-rM^4_{oooo}(x,y,z+t-1,z+t-1) \tag{44}$$

Hence, the moments can be obtained by successively solving ordinary or hyperbolic partial differential equations. Suppose for illustration that first-order moments of the initial measure are real positive constants:

$$M^1_o=m_1 \tag{45}$$

$$M^2_{oo}=m_2 \tag{46}$$

$$M^3_{ooo}=m_3 \tag{47}$$

$$M^4_{oooo}=m_4 \tag{48}$$

First Moment. For $M^1_u(t,x)$, the evolution equation (38) reduces to

$$\frac{\partial M^1_u(t,x)}{\partial t}=-\alpha M^1_u(t,x)+r(m_1-m_2) \tag{49}$$

whose solution is

$$M^1_u(t,x)=\gamma_1+[M^1_u(0,x)-\gamma_1]e^{-\alpha t}\qquad\text{where}\quad \gamma_1\equiv\frac{r(m_1-m_2)}{\alpha} \tag{50}$$

At $t=0$, from (3) and (4) we know that $v(1)=u_v(1)$. In addition, $M_o^1(t, x) \equiv M_v^1(t, x)$. Therefore, from (17)–(18),

$$M_o^1(t=0, x=1) = \int_{\mathscr{C}} v(1)\, d\mu_0 = \int_{\mathscr{C}} u_v(1)\, d\mu_0(v) = M_u^1(t=0, x=1)$$

and from the initial condition (45) we conclude $M_u^1(t=0, x) = m_1$. Hence

$$M_u^1(t, x) = \gamma_1 + [m_1 - \gamma_1] e^{-\alpha t} \tag{51}$$

Second Moments. To obtain expressions for M_{uv}^2, M_{vu}^2, and M_{uu}^2 we have to solve their respective equations of motion (remember that M_{vv}^2 is given). We first tackle (41) [this choice is arbitrary; (40) can dealt with in the same manner]:

$$\frac{\partial M_{uv}^2(t, x, y)}{\partial t} = \frac{\partial M_{uv}^2(t, x, y)}{\partial y} - \alpha M_{uv}^2(t, x, y) + r(m_2 - m_3) \tag{52}$$

with initial condition $M_{uv}^2(0, x, y) = M_{vv}^2(0, x, y) \equiv m_2$ for all x, y in the domains defined in (41). This initial condition is, as for the first moment, obtained from Eqs. (17)–(18). Equation (52) is solved using the method of characteristics, and the solution is

$$M_{uv}^2(t, x, y) = \gamma_2 + [m_2 - \gamma_2] e^{-\alpha t} \qquad \text{where} \quad \gamma_2 \equiv \frac{r(m_2 - m_3)}{\alpha} \tag{53}$$

The moment $M_{vu}^2(t, x, y)$ can be obtained in a similar way and the result is

$$M_{vu}^2(t, x, y) = M_{uv}^2(t, x, y) \tag{54}$$

This equality is due to the fact that the moments of the initial measure are constant. Finally, it is necessary to solve (43) and (44) before obtaining M_{uu}^2. Using (46)–(48),

$$M_{ou}^2 = \gamma_2 + [m_2 - \gamma_2] e^{-\alpha t} \tag{55}$$

$$M_{uo}^2 = \gamma_2 + [m_2 - \gamma_2] e^{-\alpha t} \tag{56}$$

$$M_{oou}^3 = \gamma_3 + [m_3 - \gamma_3] e^{-\alpha t} \qquad \text{where} \quad \gamma_3 \equiv \frac{r(m_3 - m_4)}{\alpha} \tag{57}$$

$$M_{uoo}^3(t) = \gamma_3 + [m_3 - \gamma_3] e^{-\alpha t} \tag{58}$$

so that the evolution for M_{uu}^2 becomes

$$\frac{\partial M_{uu}^2(t, x, y)}{\partial t} = -2\alpha M_{uu}^2(t, x, y) + 2r\gamma_2 - 2r\gamma_3 + 2re^{-\alpha t}[m_2 - m_3 - \gamma_2 + \gamma_3] \tag{59}$$

The above is a linear first-order ODE which can be solved to give

$$M_{uu}^2(t) = \frac{2r}{\alpha} e^{-\alpha t} [(m_2 - m_3) - (\gamma_2 - \gamma_3)] + \frac{r}{\alpha} (\gamma_2 - \gamma_3) + \mathscr{K} e^{-2\alpha t} \tag{60}$$

where

$$\mathscr{K} \equiv \frac{-2r}{\alpha} \left[(m_2 - m_3) - \frac{1}{2} (\gamma_2 - \gamma_3) \right] + m_2$$

This analysis can be carried out in a similar way when the moments are not constants, but such that the equations derived above remain solvable analytically.

3.3. Invariant Measures

It is of physical interest to investigate the constraint to be satisfied by a measure μ_*, invariant under the action of a differential delay equation. For the nonlinear DDE (36), the characteristic function $\mathscr{Y}$ of such a measure is defined as

$$\mathscr{Y}[J_1] = \int_{\mathscr{C}} \exp\left[i \int_0^1 J_1(x)\, u_v(x+t)\, dx \right] d\mu_* \tag{61}$$

and so we have

$$\mathscr{Y}[J_1] = \mathscr{Z}_t[J_1, 0] \qquad \text{for all } t$$

where $\mathscr{Z}_t[J_1, J_2]$ is given by (5c). The Hopf equation (16) becomes

$$\begin{aligned} 0 = & \int_0^{1-t} J_1(x) \frac{\partial}{\partial x} \left(\frac{\delta \mathscr{Y}}{\delta J_1(x)} \right) dx - \alpha \int_{1-t}^1 J_1(x) \frac{\delta \mathscr{Y}}{\delta J_1(x)}\, dx \\ & + r \int_{1-t}^1 J_1(x) \frac{\delta \mathscr{Y}}{\delta J_1(x+t-1)}\, dx \\ & - r i^{-1} \int_{1-t}^1 J_1(x) \frac{\delta^2 \mathscr{Y}}{\delta J_1^2(x+t-1)}\, dx, \qquad \forall t \end{aligned} \tag{62}$$

By choosing $t = 0$, the first integral in the Hopf equation must vanish, so that we have

$$\frac{\partial}{\partial x} \left(\frac{\delta \mathscr{Y}}{\delta J_1(x)} \right) = 0 \qquad \text{a.e.} \tag{63}$$

From this relation a *necessary* condition for the invariant measure follows: using (26), the moments must satisfy

$$\sum_{k=1}^{n} \frac{\partial}{\partial x_k} M^{n}_{u^q v^{(n-q)}}(x_1,\dots,x_k)=0 \tag{64}$$

4. THE HOPF EQUATION AND THE KRAMERS–MOYAL EXPANSION

Our treatment of delayed dynamics is developed in the spirit of statistical mechanics and ergodic theory. The need for such a treatment arises from the nature of some experimental data available in the biological sciences, where large collections of units whose individual dynamics are given by DDEs have been considered.[8]

One of the powerful tools of modern statistical mechanics is the use of equations describing the evolution of densities of initial conditions under the action of a finite-dimensional dynamical system. When that system is a set of ODEs the evolution equation is known as the *generalized Liouville equation*.[4] When the system is a set of stochastic differential equations perturbed by realizations of a Wiener process, the evolution of densities is given by the *Fokker–Planck equation*.[11] In general, for finite-dimensional systems the evolution of densities is governed by the *Kramers–Moyal (KM) equation*. It is of some interest to understand how the KM formalism carries over to systems with an infinite number of degrees of freedom such as DDEs.

The Hopf equation is probabilistic in the sense that it describes a set DDEs in the same way that the Schrödinger equation describes a microscopic physical system.[5] Given that this is precisely the role of the KM expansion for finite-dimensional dynamical systems, it is important to clarify the relation between the functional version of the KM equation and the Hopf-like equation (13) derived here. Our derivation of the functional version of the KM expansion is inspired by the derivation of the n-dimensional case given by Risken.[11]

[4] The generalized Liouville equation discussed here does not require the assumption of incompressibility. When this incompressibility assumption is valid, which is the case when dealing with conservative systems, the *generalized* Liouville equation reduces to the Liouville equation. The Liouville equation studied in Hamiltonian mechanics is a special case of a more general equation of evolution for the phase-space densities of dynamical systems. For details, see ref. 4 and references therein.

[5] A Schrödinger equation can always be transformed into a Fokker–Planck equation, which is just a truncation of the KM expansion, but the physical interpretation of the transformation remains unclear.[11]

To make the connection between $\mathscr{Z}_t$ and the KM expansion more explicit, consider the expansion (24) of the characteristic functional $\mathscr{Z}_t$. Using Eq. (26), we find that (24) becomes

$$\mathscr{Z}_t[J_1, J_2] = \sum_{p=0}^{\infty} \sum_{q=0}^{p} \int_0^1 \cdots \int_0^1 \frac{i^p}{p!} M^p_{u^q v^{(p-q)}}(t, \mathbf{x}) \times \left(\prod_{j=1}^{q} J_1(x_j)\, dx_j \right) \left(\prod_{j=q+1}^{p} J_2(x_j)\, dx_j \right) \tag{65}$$

Let $\mathscr{P}[G, H \mid G', H']$ be the transition probability functional that given the pair $(G'(x+t),\ H'(x+t))$ in $\mathscr{C} \times \mathscr{C}$, with $x \in [0, 1-t] \times (1-t, 1]$, we obtain the pair $(G(x+t+t_*),\ H(x+t+t_*))$ for $x \in [0, 1-(t+t_*)] \times (1-(t+t_*), 1]$ [i.e., G (G') is an initial function which generates a solution H (H')].

$\mathscr{W}[v, \mathscr{S}_{(t+t_*)} v]$ is related to the transition probability functional $\mathscr{P}$ by

$$\mathscr{W}[v, \mathscr{S}_{(t+t_*)} v] = \int_{\mathscr{C}} \mathscr{P}[v, \mathscr{S}_{(t+t_*)} v \mid v', \mathscr{S}_t v']\, \mathscr{W}[v', \mathscr{S}_t v']\, d\mu_0(v') \tag{66}$$

In addition, $\mathscr{W}$ is the inverse Fourier transform of the characteristic functional $\mathscr{Z}_t[J_1, J_2]$ introduced in Eq. (5c),

$$\mathscr{W}[v, \mathscr{S}_t v] = \int_{\mathscr{C}} \Upsilon^{-1} \mathscr{Z}_t[J_1, J_2]\, d\mathscr{V}[J_1, J_2] \tag{67}$$

Also,

$$\mathscr{Z}_{t+t_*} = \int_{\mathscr{C}} \tilde{\Upsilon}^{-1} \mathscr{P}[v, \mathscr{S}_{(t+t_*)} v \mid v', \mathscr{S}_t v']\, d\mu_0(v')$$

and therefore

$$\mathscr{P}[v, \mathscr{S}_{(t+t_*)} v \mid v', \mathscr{S}_t v'] = \int_{\mathscr{C}} \tilde{\Upsilon} \mathscr{Z}_{t+t_*}[J_1, J_2]\, d\mathscr{V}[J_1, J_2] \tag{68}$$

where

$$\tilde{\Upsilon} = \exp\left(-i \left\{ \int_0^1 J_2(x)[v(x) - v'(x)]\, dx + \int_0^1 J_1(x)[-\mathscr{S}_t v'(x) + \mathscr{S}_{(t+t_*)} v(x)]\, dx \right\} \right)$$

and the measure of integration $\mathscr{V}[J_1, J_2]$ is a measure like the one used in the definition (5a) of the characteristic functional. More precisely, *the measure* $\mathscr{W}$ *describes the distribution of functions in* $\mathscr{C}$ *generating pairs* $(v, \mathscr{S}_t v)$ *under the action of the transformation* (5b), *and the measure* $\mathscr{V}[J_1, J_2]$ *describes the distribution of functions generating pairs* (J_1, J_2) *in the same space.* Inserting (65) into Eq. (68), we obtain

$$\begin{aligned}
&\mathscr{P}[v, \mathscr{S}_{(t+t_*)} v \mid v', \mathscr{S}_t v'] \\
&\quad = \int_{\mathscr{C}} \tilde{\Upsilon} \left[\sum_{p=0}^{\infty} \sum_{q=0}^{p} \int_0^1 \cdots \int_0^1 \frac{1}{p!} M^p_{u^q v^{(p-q)}}(t + t_*, x) \right. \\
&\qquad \left. \times \left(\prod_{j=1}^{q} iJ_1(x_j)\, dx_j \right) \left(\prod_{j=q+1}^{p} iJ_2(x_j)\, dx_j \right) \right] d\mathscr{V}[J_1, J_2] \qquad (69)
\end{aligned}$$

The Dirac δ functional is a straightforward generalization of the more usual N-dimensional version. It satisfies

$$\int_{\mathscr{C}} \delta[H - G]\, d\omega = \begin{cases} 1 & \text{if } \quad H = G \text{ almost everywhere} \\ 0 & \text{otherwise} \end{cases} \qquad (70)$$

where H and G are elements of $\mathscr{C}$, ω is a measure defined on $\mathscr{C}$, and the result of the integration is a number (not a function). We will use this definition to simplify expansion (69). Before doing so, recall the following identity (see the Appendix):

$$\begin{aligned}
&\int_{\mathscr{C}} \left(\prod_{j=1}^{q} iJ_1(x_j) \right) \left(\prod_{j=q+1}^{p} iJ_2(x_j) \right) \tilde{\Upsilon}\, d\mathscr{V}[J_1, J_2] \\
&\quad = \left(\frac{(-\delta)^p}{\delta u_v(x_1) \cdots \delta u_v(x_q)\, \delta v(x_{(q+1)}) \cdots \delta v(x_p)} \right) \delta[v - v']\, \delta[\mathscr{S}_{(t+t_*)} v - \mathscr{S}_t v'] \\
&\qquad\qquad (71)
\end{aligned}$$

Introduce the symbolic differential operator

$$\mathscr{Q}(u_v^q, v^{(p-q)}) = \frac{(-\delta)^p}{\delta u_v(x_1) \cdots \delta u_v(x_q)\, \delta v(x_{(q+1)}) \cdots \delta v(x_p)}$$

Using the identity (71), we have that Eq. (69) reduces to

$$\begin{aligned}
&\mathscr{P}[v, \mathscr{S}(t + t_*) v \mid v', \mathscr{S}_t v'] \\
&\quad = \left[\sum_{p=0}^{\infty} \sum_{q=0}^{p} \frac{1}{p!} \int_0^1 \cdots \int_0^1 \mathscr{Q}(u_v^q, v^{(p-q)})\, M^p_{u^q v^{(p-q)}}(t + t_*, \mathbf{x})\, d\mathbf{x} \right] \\
&\qquad \times \delta[v - v']\, \delta[\mathscr{S}_{(t+t_*)} v - \mathscr{S}_t v'] \qquad (72)
\end{aligned}$$

Suppose that the moments $M^{p}_{u^q v^{(p-q)}}(t+t_*, \mathbf{x})$ can be expanded in a power series about $t_*=0$:

$$\frac{1}{p!} M^{p}_{u^q v^{(p-q)}}(t+t_*, x) = \mathscr{B}^{p}_{u^q v^{(p-q)}}(t, \mathbf{x}) t_* + \mathcal{O}(t_*^2) + \cdots \tag{73}$$

Equation (72) with expansion (73) is inserted in (66) to yield

$$\begin{aligned} &\mathscr{W}[v, \mathscr{S}_{(t+t_*)}v] \\ &\quad \simeq \int_{\mathscr{C}} \left[\sum_{p=0}^{\infty} \sum_{q=0}^{p} \int_0^1 \cdots \int_0^1 \mathscr{Q}(u_v^q, v^{(p-q)})\, \mathscr{B}^{p}_{u^q v^{(p-q)}}(t, \mathbf{x}) t_*\, d\mathbf{x} \right] \\ &\qquad \times \mathscr{W}[v', \mathscr{S}_t v']\, \delta[v-v')\, \delta[\mathscr{S}_{(t+t_*)}v - \mathscr{S}_t v']\, d\mu_0(v') \end{aligned} \tag{74}$$

where the measure μ_0 is defined on the space of initial functions. Carrying out the functional integration in (74) and dividing by t_* gives

$$\begin{aligned} &\frac{\mathscr{W}[v, \mathscr{S}_{(t+t_*)}v] - \mathscr{W}[v, \mathscr{S}_t v]}{t_*} \\ &\quad \simeq \left[\sum_{p=1}^{\infty} \sum_{q=0}^{p} \int_0^1 \cdots \int_0^1 \mathscr{Q}(u_v^q, v^{(p-q)})\, \mathscr{B}^{p}_{u^q v^{(p-q)}}(t, \mathbf{x})\, d\mathbf{x} \right] \\ &\qquad \times \mathscr{W}[v, \mathscr{S}_{(t+t_*)}v] \end{aligned} \tag{75}$$

Taking the limit $t_* \to 0$, we get the *infinite-dimensional version of the Kramers–Moyal expansion*:

$$\begin{aligned} &\frac{\partial \mathscr{W}[v, \mathscr{S}_t v]}{\partial t} \\ &\quad \simeq \left[\sum_{p=1}^{\infty} \sum_{q=0}^{p} \int_0^1 \cdots \int_0^1 \mathscr{Q}(u_v^q, v^{(p-q)})\, \mathscr{B}^{p}_{u^q b^{(p-q)}}(t, \mathbf{x})\, d\mathbf{x} \right] \mathscr{W}[v, \mathscr{S}_t v] \end{aligned} \tag{76}$$

The above analysis is not restricted to delay differential equations of the form (1a), (1b). The only real constraint imposed on the dynamical system under consideration is that its phase space be a normed function space. Therefore this analysis is also valid for the statistical investigation of partial differential equations. In fact this approach was pioneered by Hopf in ref. 2, in which he derived an evolution equation for the characteristic functional describing the solutions of the Navier–Stokes equations statistically.

From (67), it is clear that taking the Fourier transform of (76) yields the evolution equation for the characteristic functional $\mathscr{Z}_t[J_1, J_2]$. However, in Section 2.1, we derived the Hopf evolution equation (13) for

$\mathscr{Z}_t[J_1, J_2]$. Thus we conclude that the Hopf equation (13) is the Fourier transform of the infinite-dimensional extension of the Kramers–Moyal expansion (76).

From (73), the KM coefficients are given by solving the partial differential equations presented in Section 3.2.

5. DISCUSSION

The introduction of the joint characteristic functional (5c) provides a tool for the investigation of differential delay equations from a probabilistic point of view. This approach is meaningful from a physical perspective when dealing with large collections of entities whose dynamics are governed by DDEs. For example, it is well known that certain aspects of neuronal activity can be described with nonlinear DDEs of the type discussed here.[7] In addition, physiological evidence suggests that in some cases the functional unit in the brain is not the single neuron, but a collection of neurons. Therefore, it is expected that to analyze physiologically plausible neural networks a probabilistic approach will be more adequate than a purely deterministic one. Moreover, a probabilistic description is clearly needed when the models are formulated as stochastic DDEs. In this case, the characteristic functional (5c) is no longer valid, but it can be modified in a way similar to that presented in Section 7 of ref. 5 (in the context of stochastic PDEs), and a *three-interval characteristic functional* should then be considered.

As a conclusion, we note that the expansion (24) is similar to functional expansions in quantum field theory and statistical mechanics which are treated in a perturbative manner and analyzed with Feynman diagrams. Although the moment PDEs of Section 3.2 can indeed be deduced from a graphical analysis of expansion (24),[6] it remains to be seen whether the introduction of proper Feynman diagrams will provide, through graphical manipulations, significant insight into delayed dynamics.

6. SUMMARY

In this paper, we have derived a Hopf-like functional equation for the evolution of the characteristic functional of a measure defined on the space of initial functions for a class of nonlinear differential delay equations. Using perturbation theory, the Hopf equation is reduced to an infinite chain of partial differential equations for the moments of the evolving distributions of functions. The first two moments are obtained explicitly for a DDE with a quadratic nonlinearity, when the moments of the initial measure are constant. Finally, we show that the Hopf equation is the Fourier transform of the infinite-dimensional version of the Kramers–Moyal expansion.

APPENDIX

Here we derive Eq. (71),

$$\int_{\mathscr{C}} \tilde{\Upsilon}\left(\prod_{j=1}^{q} iJ_1(x_j)\right)\left(\prod_{j=q+1}^{p} iJ_2(x_j)\right) d\mathscr{V}[J_1, J_2]$$
$$= \left(\frac{(-\delta)^p}{\delta u(x_1)\cdots\delta u(x_q)\,\delta v(x_{(q+1)})\cdots\delta v(x_p)}\right)\delta[v-v']\,\delta[\mathscr{S}_{(t+t_*)}v-\mathscr{S}_t v'] \tag{A1}$$

Recall that the δ functional can be written

$$\delta[G-H] = \int_{\mathscr{C}} \exp\left\{-i\int_0^1 K(r)[G(r)-H(r)]\,dr\right\} d\omega$$
$$\equiv \int_{\mathscr{C}} \Upsilon_1\, d\omega \tag{A2}$$

where the functions H and G, defined on $r\in[0,1]$, are elements of the function space $\mathscr{C}([0,1])$. Functionally differentiating (A2) yields

$$\frac{\delta\delta[G-H]}{\delta G} = \int_{\mathscr{C}} [-iK(r)]\,\Upsilon_1\, d\omega \tag{A3}$$

More generally,

$$\frac{\delta^q\delta[G-H]}{\delta G(r_1)\cdots\delta G(r_q)} = \int_{\mathscr{C}} (-i)^q\left(\prod_{j=1}^{q} K(r_j)\right)\Upsilon_1\, d\omega \tag{A4}$$

We can define $\delta[E-F]$, where E and F are elements of $\mathscr{C}[0,1]$, is a fashion analogous to (A2):

$$\delta[E-F] = \int_{\mathscr{C}} \exp\left\{-i\int_0^1 L(r)[E(r)-F(r)]\,d\tau\right\} d\omega$$
$$\equiv \int_{\mathscr{C}} \Upsilon_2\, d\omega \tag{A5}$$

From (A4), it is clear that if $\delta[G-H]$ is differentiated q times while $\delta[E-F]$ is differentiated $(p-q)$ times, the product of the two quantities will be

$$\frac{\delta^q \delta[G-H]}{\delta G(r_1)\cdots\delta G(r_q)} \times \frac{\delta^{(p-q)}\delta[E-F]}{\delta E(r_{q+1})\cdots\delta E(r_p)}$$
$$= \frac{\delta^p}{\delta G(r_1)\cdots\delta G(r_q)\,\delta E(r_{(q+1)})\cdots\delta E(r_p)}\,\delta[G-H]\,\delta[E-F]$$
$$= \int_{\mathscr{C}} \left(\prod_{j=1}^{q} -iK(r_j)\,dr\right)\left(\prod_{j=q+1}^{p} -iL(r_j)\right) \Upsilon_1 \Upsilon_2\, d\mathscr{V}[J_1, J_2] \tag{A6}$$

Replacing $[G-H]$ by $[v-v']$ and $[E-F]$ by $[\mathscr{S}_{(t+t_*)}v - \mathscr{S}_t v']$, identifying the source functions K, LM with J_1, J_2, and using the identity $\Upsilon_1 \Upsilon_2 = \tilde{\Upsilon}$, (A6) yields (71).

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